Rule of nine for the recursive digit sum

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1 Recursive digit sum

In the following let the numbers a_k be the coefficients of the base ten representation of the natural number n, $n = \sum_{k=0}^{t} a_k 10^k$.

Definition 1.1 (Digit sum) *The **digit sum** of n, $\sigma(n)$, is defined as*

$$\sigma(n) := \sum_{k=0}^{t} a_k$$

Definition 1.2 (Recursive digit sum) *The **Recursive digit sum** of n, $\tau(n)$, is defined as*

$$\begin{aligned} \tau(n) &= n && \text{for } n \leq 9 \\ \tau(n) &= \tau(\sigma(n)) && \text{for } n > 9 \end{aligned}$$

We have to show, that the above definition is well defined. This is done by showing, that for any n, $\tau(n)$ converges to a number between zero and nine.
First we show that the argument in the recursive call to τ is strictly decreasing.

Lemma 1.1 (Argument to τ is strictly decreasing) *The recursive argument $\sigma(n)$ to τ is strictly decreasing, such that for $n > 9$ the following inequality is valid*

$$\sigma(n) < n$$

<u>*Proof*</u>
The proof is by induction on the maximum power of ten, t, in the base 10 representation of n. So we want to prove that $\sum_{k=0}^{t} a_k < \sum_{k=0}^{t} a_k 10^k$. Let $P(t)$ be the proposition that

$$\sum_{k=0}^{t} a_k < \sum_{k=0}^{t} a_k 10^k$$

<u>"$P(1)$ is true"</u>

$$\begin{aligned} \sum_{k=0}^{t} a_k &= a_0 + a_1 \\ &< a_0 10^0 + a_1 10^1 \\ &= \sum_{k=0}^{t} a_k 10^k \end{aligned}$$

<u>"$P(t)$ implies $P(t+1)$"</u> Assume, that the claim is valid for all numbers with a maximum power equal to t. We prove that this implies that the claim is valid for all numbers

with a maximum power equal to $t+1$.

$$\begin{aligned}
\sum_{k=0}^{t+1} a_k &= \sum_{k=0}^{t} a_k + a_{t+1} \\
&< \sum_{k=0}^{t} a_k 10^k + a_{t+1} \\
&< \sum_{k=0}^{t} a_k 10^k + a_{t+1} 10^{t+1} \\
&= \sum_{k=0}^{t+1} a_k 10^k
\end{aligned}$$

□

Proposition 1.1 (Convergence of τ)
The recursive digit sum $\tau(n)$ converges to a number between zero and nine.

<u>Proof</u>
<u>"$n \leq 9$"</u>: Then, by definition, $\tau(n)$ is equal to n.
<u>"$n > 9$"</u>: Then the lemma shows, that the following inequality is valid:

$$\sigma(n) < n$$

By definition, τ is applied to the strictly smaller (non-negative) number $\sigma(n)$. □

In order to show some propositions on the recursive digit sum with respect to divisibility by nine, consider the following table:

$\frac{0}{9} = 0,00000...$

$\frac{1}{9} = \frac{1 \cdot 10^1}{9} \cdot 10^{-1} = \frac{10}{9} \cdot 10^{-1} = \frac{9+1}{9} \cdot 10^{-1} = (1 + \frac{1}{9}) \cdot 10^{-1} = 0,11111...$

$\frac{2}{9} = \frac{2 \cdot 10^1}{9} \cdot 10^{-1} = \frac{20}{9} \cdot 10^{-1} = \frac{18+2}{9} \cdot 10^{-1} = (2 + \frac{2}{9}) \cdot 10^{-1} = 0,22222...$

$\frac{3}{9} = \frac{3 \cdot 10^1}{9} \cdot 10^{-1} = \frac{30}{9} \cdot 10^{-1} = \frac{27+3}{9} \cdot 10^{-1} = (3 + \frac{3}{9}) \cdot 10^{-1} = 0,33333...$

$\frac{4}{9} = \frac{4 \cdot 10^1}{9} \cdot 10^{-1} = \frac{40}{9} \cdot 10^{-1} = \frac{36+4}{9} \cdot 10^{-1} = (4 + \frac{4}{9}) \cdot 10^{-1} = 0,44444...$

$\frac{5}{9} = \frac{5 \cdot 10^1}{9} \cdot 10^{-1} = \frac{50}{9} \cdot 10^{-1} = \frac{45+5}{9} \cdot 10^{-1} = (5 + \frac{5}{9}) \cdot 10^{-1} = 0,55555...$

$\frac{6}{9} = \frac{6 \cdot 10^1}{9} \cdot 10^{-1} = \frac{60}{9} \cdot 10^{-1} = \frac{54+6}{9} \cdot 10^{-1} = (6 + \frac{6}{9}) \cdot 10^{-1} = 0,66666...$

$\frac{7}{9} = \frac{7 \cdot 10^1}{9} \cdot 10^{-1} = \frac{70}{9} \cdot 10^{-1} = \frac{63+7}{9} \cdot 10^{-1} = (7 + \frac{7}{9}) \cdot 10^{-1} = 0,77777...$

$\frac{8}{9} = \frac{8 \cdot 10^1}{9} \cdot 10^{-1} = \frac{80}{9} \cdot 10^{-1} = \frac{72+8}{9} \cdot 10^{-1} = (8 + \frac{8}{9}) \cdot 10^{-1} = 0,88888...$

$\frac{9}{9} = 1,00000...$

As can be seen from the table, dividing the coefficients a_k by nine gives a fraction of an inifinite period of one, like for example $\frac{4}{9}$ with a fraction of an infinite period of one with the value 4 $(0,44444...)$. But for $n = \sum_{k=0}^{t} a_k 10^k$ we also have to consider the multiplication of a_k by a power of 10 when dividing by nine. Intuitively, we see that $\frac{a_k 10^k}{9}$ is an integer part and a fraction with the same value as $\frac{a_k}{9}$. We now state and prove this formally.

Lemma 1.2 (Rule of nine for the form of multiplas of powers of ten)
For any integer a with $0 \leq a \leq 9$ and for all $k \geq 0$ we can write $\frac{a}{9}10^k$ on the following form:

$$\frac{a}{9}10^k = \sum_{r=1}^{k} a10^{k-r} + \frac{a}{9}$$

<u>*Proof*</u>
For any non-negative number k let the proposition $P(k)$ be that

$$\frac{a}{9}10^k = \sum_{r=1}^{k} a10^{k-r} + \frac{a}{9}$$

We prove the lemma by induction on k.

"$P(0)$ is true"
For $k := 0$ we have

$$\frac{a}{9}10^k = \frac{a}{9}10^0 = \frac{a}{9}1 = \frac{a}{9} = \sum_{r=1}^{0} a10^{0-r} + \frac{a}{9}$$

The sum $\sum_{r=1}^{0} a10^{0-r}$ has no terms but is included in order to show the validity of the form.

"$P(1)$ is true"
For $k := 1$ we have

$$\frac{a}{9}10^k = \frac{a}{9}10^1 = a\frac{10}{9} = a\left(\frac{9+1}{9}\right) = a(1+\frac{1}{9}) = a10^0 + \frac{a}{9} = \sum_{r=1}^{k} a10^{k-r} + \frac{a}{9}$$

"$P(k)$ implies $P(k+1)$" Assume that $P(k)$ is true. We then have

$$\begin{aligned}
\frac{a}{9}10^{k+1} &= \left(\frac{a}{9}10^k\right)\cdot 10 \\
&= \left(\sum_{r=1}^{k} a10^{k-r} + \frac{a}{9}\right)\cdot 10 \\
&= 10\cdot\sum_{r=1}^{k} a10^{k-r} + \frac{10a}{9} \\
&= 10\cdot\sum_{r=1}^{k} a10^{k-r} + \frac{9a+a}{9} \\
&= \sum_{r=1}^{k} a10^{k+1-r} + a10^0 + \frac{a}{9} \\
&= \sum_{r=1}^{k+1} a10^{k+1-r} + \frac{a}{9}
\end{aligned}$$

This proves $P(k+1)$. □

Lemma 1.3 (Rule of nine for the digit sum)
For any natural number n we have that $9|\sigma(n)$ if and only if $9|n$.

Proof
For $n = \sum_{k=1}^{t} a_k 10^k$ we have that $\sigma(n) = \sum_{k=1}^{t} a_k$. Define z as $\sigma(n)$ divided by 9:

$$z := \frac{\sigma(n)}{9} = \frac{\sum_{k=1}^{t} a_k}{9} = \sum_{k=1}^{t} \frac{a_k}{9}$$

Now, for $\frac{n}{9}$, we have the following:

$$\begin{aligned}
\frac{n}{9} &= \frac{\sum_{k=1}^{t} a_k 10^k}{9} \\
&= \sum_{k=1}^{t} \frac{a_k 10^k}{9} \\
&= \sum_{k=1}^{t} \left(\sum_{r=1}^{k} a_k 10^{k-r} + \frac{a_k}{9} \right) && \textit{by the lemma for the form} \\
&= \sum_{k=1}^{t} \left(\sum_{r=1}^{k} a_k 10^{k-r} \right) + \sum_{k=1}^{t} \frac{a_k}{9} \\
&= \sum_{k=1}^{t} \left(\sum_{r=1}^{k} a_k 10^{k-r} \right) + z
\end{aligned}$$

Note, that the sum $\sum_{k=1}^{t} \left(\sum_{r=1}^{k} a_k 10^{k-r} \right)$ is an integer, so divisibility of n by 9 depends on whether z is an integer or a fraction. Using this observation, we can now prove the two claimed implications.

"<u>$9|\sigma(n) \implies 9|n$</u>": Assume that $9|\sigma(n)$. Then z is an integer and therefore $\frac{n}{9}$ is an integer and so we have, that $9|n$.

"<u>$9|n \implies 9|\sigma(n)$</u>": Assume that $9|n$. Then z is an integer and therefore $\frac{\sigma(n)}{9}$ is an integer and so we have, that $9|\sigma(n)$. □

Proposition 1.2 (Rule of nine for the recursive digit sum)
For any natural number n we have that $\tau(n) = 9$ if and only if $9|n$.

<u>*Proof*</u>

"<u>$9|n \implies \tau(n) = 9$</u>": Assume that $9|n$. Then, by the lemma, $9|\sigma(n)$. So every argument in each of the recursive calls to τ is divisible by 9. Therefore $\tau(n)$ converges to a number divisible by 9. This number must be 9 and hence $\tau(n) = 9$.

"<u>$\tau(n) = 9 \implies 9|n$</u>": Assume that $\tau(n) = 9$. If $n \leq 9$ then n must be equal to 9 so $9|n$. This proves the proposition for $n \leq 9$. For $n > 9$ we have, that $\tau(n) = \tau(\sigma(n))$. We then use induction over the number of recursive calls, r, denoted by the subscript in τ_r.

Let $P(r)$ be the proposition that $\tau_r(n) = 9$ implies $9|n$.

"<u>$P(1)$ is true</u>": For $r = 1$ we have that $\tau_1(n) = \tau_0(\sigma(n)) = 9$. So $\sigma(n)$ must be equal to 9 and therefore we have that $9|\sigma(n)$ and by the lemma we finally get that $9|n$. This proves that $P(1)$ is true.

"$P(r)$ implies $P(r+1)$"': Assume, that for r recursive calls we have that if $\tau_r(n) = 9$ then $9|n$. For $r+1$ recursive calls assume that $\tau_{r+1}(n) = \tau_r(\sigma(n)) = 9$. By induction we can conclude that $9|\sigma(n)$ and then, by the lemma, we can conclude that $9|n$. This proves the induction step.

In all the proposition has been proved. □